1·0
2·8
2·5
5·0 3·15
2·2
3·5
2·0
4·0
4·5
1·8
1·1
1·25 1·4
1·6

NATIONAL BUREAU OF STANDARDS
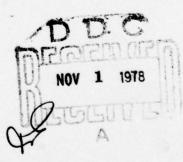MICROCOPY RESOLUTION TEST CHART

ADA060623

MRC Technical Summary Report #1866

REMARKS ON NONLINEAR ERGODIC THEORY
IN HILBERT SPACE

A. Pazy

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

July 1978

(Received June 6, 1978)

78 09 12 033

(6) REMARKS ON NONLINEAR ERGODIC THEORY IN HILBERT SPACE.

(10) A. Pazy

(11) Jul 78

(14) MRC-TSR-1866

(12) 18 p.

ABSTRACT

In this note we give a simple unified presentation of some recent ergodic results for semigroups of nonexpansive mappings in Hilbert space.

(15) DAAG29-75-C-0024, NSF-MCS78-01245

sh

231 200

SIGNIFICANCE AND EXPLANATION

The classical ergodic theory originates from statistical mechanics. There, one is interested in the existence of limits of certain time averages of functions of the state of a given deterministic (mechanical) system. Formulated mathematically the problem reduces to the question of existence of limits of certain averages of iterates of linear operators in the discrete case or averages of semigroups of linear operators in the continuous case.

Recently, similar questions have been studied for some classes of nonlinear operators. The results that were obtained are rather elegant and they shed some new light on the classical linear results. The physical significance of these nonlinear results however is not clear.

In this note we present a simple unified treatment of some of these recent "nonlinear ergodic" theorems.

REMARKS ON NONLINEAR ERGODIC THEORY IN HILBERT SPACE

A. Pazy

1. Introduction

The recent developments in the ergodic theory of nonlinear mappings in Hilbert space started with the result of B. Baillon [1]. Baillon considered a nonexpansive map T of a real Hilbert space H into itself. He proved that if T has fixed points in H then for every $x \in H$, the Cesaro means

$$(1.1) \qquad \sigma_n(x) = \frac{1}{n} \sum_{j=0}^{n-1} T^j x$$

converge weakly as $n \to \infty$ to a fixed point of T. A corresponding theorem for a strongly continuous one parameter semigroup of nonexpansive mappings $S(t)$, $t \geq 0$ was given soon after Baillon's work by B. Baillon and H. Brezis [2]. The proof of the result for the continuous case is much simpler than Baillon's proof for the discrete case. This is mainly due to the use of the generation theory of strongly continuous semigroups of nonexpansive mappings in Hilbert space.

Brezis and Browder [6] extended the original result of Baillon to the case of more general averages. Similar results were also obtained by R. Bruck [9] and S. Reich [13]. The corresponding version for strongly continuous semigroups of nonexpansive mappings was derived by H. Brezis [4], using the generation theory for such semigroups. S. Reich [17] derived the same result by reducing the continuous parameter case to the discrete parameter case.

The purpose of this note is to present the above mentioned results in a unified simple way. The main property of nonexpansive mappings that is used is that weak limits of averages of iterates of such mapping are fixed points of all iterates of these mappings (see lemma 3.3 below). This feature of nonexpansive mappings together with some simple facts about weak convergence in Hilbert space, yield the ergodic results.

Section 2 is devoted to some elementary considerations concerning the weak convergence of bounded functions from the reals into a real Hilbert space H. The main result of this section is proposition 2.3 from which some useful sufficient conditions for weak convergence are derived. These conditions in turn, are used in section 3 to obtain a general version of the ergodic theorem, theorem 3.4, for nonexpansive mappings. Theorem 3.4 contains as special cases the results of [1], [6], [9], [16] in the discrete case and of [2], [4], [17] in the continuous case. For a similar result in a more general setup see also Brezis-Browder [7]. In the proof of theorem 3.4 no use is made of the generation theory of strongly continuous semigroups of nonexpansive mappings. The same proof works in both the discrete and continuous parameter cases. Therefore, as can be expected, the continuity in the parameter $t \geq 0$, of $S(t)x$ is not necessary for the ergodic theorem to hold in the continuous parameter case.

In the last section, section 4, we use the results of section 2 to derive some recent ergodic results due to P. L. Lions [12] for the products of resolvents of maximal monotone operators in Hilbert space.

2. Weak convergence

Let H be a real Hilbert space and let $a(t):D \subset \mathbb{R}^+ \to H$ be a fixed bounded function. The domain D of $a(t)$ will be always assumed to be an unbounded subset of $\mathbb{R}^+$. With the function $a(t)$ we associate the functional

$$(2.1) \qquad F(v) = \limsup_{\substack{t \to \infty \\ t \in D}} \|a(t) - v\|^2$$

In the rest of this note we will consider limits of $a(t)$ as $t \to \infty$ without explicitly stating that $t \to \infty$ in D. The functional $F(v)$ defined above is locally lipschitz continuous and strictly convex. The strict convexity follows from the identity

$$(2.2) \quad \|a - \alpha u - (1 - \alpha)v\|^2 = \alpha\|a - u\|^2 + (1 - \alpha)\|a - v\|^2 - 2\alpha(1 - \alpha)\|u - v\|^2$$

which holds for all $a, u, v \in H$ and $\alpha \in \mathbb{R}$. Moreover, $F(v) \to +\infty$ as $\|v\| \to \infty$ and therefore F has a unique minimum in H. We follow M. Edelstein [11] in defining:

Definition 2.1

The unique point $a_0 \in H$ satisfying

$$(2.3) \qquad F(a_0) = \min_{u \in H} F(u)$$

is called the asymptotic center of $a(t)$ and it is denoted by $a_0 = AC(a(t))$.

We recall the definition of the weak w-limit set $W(a(t))$ of the function $a(t)$

$$(2.4) \quad W(a(t)) = \{u : u \in H, \ u = \text{w-}\lim_{t_k \to \infty} a(t_k), \ \text{for some sequence} \ \{t_k\} \subset D, \ t_k \to \infty\}$$

where w-lim denotes the weak limit in H. Denoting by $\overline{\text{conv}}\, W(a(t))$ the closed convex hull of $W(a(t))$, we have

Proposition 2.2

$$(2.5) \qquad AC(a(t)) \in \overline{\text{conv}}\, W(a(t)).$$

If moreover, $\text{w-}\lim_{t \to \infty} a(t) = a$ then $a = AC(a(t))$.

Proof:

Let $a_0 = AC(a(t))$ and let a^1 be the orthogonal projection of a_0 on $\overline{\text{conv}}\, W(a(t))$, then $(w - a^1, a^1 - a_0) \geq 0$ for every $w \in \overline{\text{conv}}\, W(a(t))$ and consequently

$$(2.6) \qquad \liminf_{t \to \infty}(a(t) - a^1, a^1 - a_0) \geq 0 .$$

-3-

Passing to the limit through an appropriate sequence of t and using (2.6) in

(2.7) $$\|a(t) - a_0\|^2 = \|a(t) - a^1\|^2 + 2(a(t) - a^1, a^1 - a_0) + \|a^1 - a_0\|^2$$

we obtain

(2.8) $$F(a_0) \geq F(a^1) + \|a^1 - a_0\|^2 .$$

From the uniqueness of the minimum of F it follows that $a_0 = a^1 \in \overline{\text{conv}}\, W(a(t))$. The second part of the proposition is an immediate consequence of the first. ∎

In our next results we will use the following notations;

(2.9) $$L = L(a(t)) = \{u : u \in H, \, \exists \lim_{t \to \infty}(a(t), u)\}$$

and

(2.10) $$N = N(a(t)) = \{u : u \in H, \, \exists \lim_{t \to \infty} \|a(t) - u\|^2\}$$

L is obviously a closed linear subspace of H and $a(t)$ converges weakly as $t \to \infty$ if and only if $L = H$. From the definition of L it follows that if $w_1, w_2 \in W(a(t))$ then $(w_1 - w_2, \ell) = 0$ for every $\ell \in L$. This result extends obviously to every $w_1, w_2 \in \overline{\text{conv}}\, W(a(t))$. The main result of this section is:

Proposition 2.3

$N(a(t)) \cap \overline{\text{conv}}\, W(a(t))$ contains at most one point. If $N(a(t)) \cap \overline{\text{conv}}\, W(a(t)) \neq \phi$ then

(2.11) $$\{AC(a(t))\} = N(a(t)) \cap \overline{\text{conv}}\, W(a(t)) .$$

Proof:

Using polarization it is obvious that if $u_1, u_2 \in N$ then $u_1 - u_2 \in L$. Therefore, by the remarks preceding this proposition, it follows that if $u_1, u_2 \in N \cap \overline{\text{conv}}\, W$ then $u_1 = u_2$ and thus $N \cap \overline{\text{conv}}\, W$ is at most a singleton. Let $\{v\} = N \cap \overline{\text{conv}}\, W$. Since $v \in N$, $F(v) = \lim_{t \to \infty} \|a(t) - v\|^2$. Let $w \in W$, $w = \text{w-}\lim_{t_k \to \infty} a(t_k)$ and let $u \in H$.

Passing to the limit as $t_k \to \infty$ in

$$\|a(t_k) - u\|^2 = \|a(t_k) - v\|^2 + 2(a(t_k) - v, v - u) + \|v - u\|^2$$

we find

(2.12) $$F(u) \geq F(v) + 2(w - v, v - u) + \|v - u\|^2$$

-4-

The inequality (2.12) holds for all $w \in W$ and therefore also for all $w \in \overline{\text{conv}} \, W$.

Since $v \in \overline{\text{conv}} \, W$ we can substitute $w = v$ in (2.12) and obtain

$$F(u) \geq F(v) + \|u - v\|^2 \qquad \text{for all } u \in H.$$

From the uniqueness of the minimum of F it follows that $v = AC(a(t))$. ∎

Corollary 2.4

Let $b(t) : D \subset \mathbb{R}^+ \to H$ be bounded. If $W(b(t)) \subset \overline{\text{conv}} \, W(a(t)) \cap N(a(t))$ then $AC(a(t)) = AC(b(t))$ and

$$(2.13) \qquad \qquad \underset{t \to \infty}{w\text{-lim}} \, b(t) = AC(a(t))$$

Proof:

Since $b(t)$ is bounded $W(b(t)) \neq \phi$ therefore $\overline{\text{conv}} \, W(a(t)) \cap N(a(t)) \neq \phi$ and by proposition 2.3 $W(b(t)) = \{AC(a(t))\}$ i.e. $\underset{t \to \infty}{w\text{-lim}} \, b(t) = AC(a(t))$. From proposition 2.2 we then have $AC(b(t)) = AC(a(t))$. ∎

Corollary 2.4 is a restatement of a generalization given by Brezis and Browder [6] to a lemma of Opial [13]. Taking $b(t) = a(t)$ in corollary 2.4 we obtain

Corollary 2.5

Let $a(t) : D \subset \mathbb{R}^+ \to H$ be bounded. If $W(a(t)) \subset N(a(t))$ then

$$(2.14) \qquad \qquad \underset{t \to \infty}{w\text{-lim}} \, a(t) = AC(a(t)) \, .$$

We conclude this section with the following consequence of proposition 2.3

Proposition 2.6

Let $a(t) : D \subset \mathbb{R}^+ \to H$ be bounded. If $N(a(t)) \neq \phi$ then $a(t)$ converges weakly as $t \to \infty$ in D if and only if

$$(2.15) \qquad \qquad \underset{t \to \infty}{w\text{-lim}} (a(t + h) - a(t)) = 0$$

for every $h \in \mathbb{R}^+$ for which $D \supset D + h = \{t + h : t \in D\}$.

Proof:

Clearly if $a(t)$ converges weakly (2.15) holds. Let $u \in N(a(t))$ and $y \in H$ then

$$(2.16) \quad \big| \, \|a(t+h) - y\|^2 - \|a(t) - y\|^2 \, \big| \leq \big| \, \|a(t+h) - u\|^2 - \|a(t) - u\|^2 \, \big|$$
$$+ 2\big|(a(t+h) - a(t), \, u - y)\big| \, .$$

Since $\|a(t) - u\|^2$ converges at $t \to \infty$, (2.16) implies the convergence of $\|a(t) - y\|^2$ as $t \to \infty$ for all $y \in H$. Consequently $N = H$ and from proposition 2.3 it follows that $W(a(t)) = AC(a(t))$. ∎

3. The Ergodic theorem for semigroups of nonexpansive mappings

Let $D \subseteq \mathbb{R}^+$ be an additive semigroup i.e. $t_1, t_2 \in D$ imply $t_1 + t_2 \in D$. The main examples that we have in mind are $D = \mathbb{R}^+$ and $D = \mathbb{Z}^+ = \{n : n \geq 0\}$, but the results apply to any additive semigroup $D \subseteq \mathbb{R}^+$. Since $D \subseteq \mathbb{R}^+$ the order in $\mathbb{R}$ induces a natural order in D. We will continue to assume that D is unbounded, and $t \to \infty$ in D will be usually written as $t \to \infty$ without explicitly stating that $t \in D$.

Let C be a closed convex subset of H. A family of mappings $S(t) : C \to C$ $t \in D$ is called a semigroup of nonexpansive mappings on C if:

$$(3.1) \qquad S(0) = I, \quad S(t + s) = S(t) \circ S(s), \quad s, t \in D$$

and

$$(3.2) \qquad \|S(t)x - S(t)y\| \leq \|x - y\| \quad \text{for all} \quad x, y \in C, \quad t \in D .$$

We denote by F the (possibly empty) set of fixed points of $S(t)$ i.e.

$$(3.3) \qquad F = \{x : x \in C, S(t)x = x \quad \text{for all} \quad t \in D\}$$

Note that in the discrete case, F coincides with the set of fixed points of $S(t_0)$ where t_0 is the smallest nonzero element of D.

From (3.2) it follows that F is a closed convex subset of C. For every $p \in F$, $x \in C$, $s, t \in D$, $t \geq s$ we have:

$$(3.4) \qquad \|S(t)x - p\| = \|S(t - s) \circ S(s)x - S(t - s)p\| \leq \|S(s)x - p\| .$$

So, if $F \neq \phi$, $t \to S(t)x$ is a bounded function of $D \subseteq \mathbb{R}^+$ into H and

$$(3.5) \qquad F \subset N(S(t)x) .$$

Combining this observation with corollary 2.5 we have

Proposition 3.1

If $F \neq \phi$ then $W(S(t)x) \subset F$ implies

$$(3.6) \qquad \underset{t \to \infty}{\text{w-lim}} \, S(t)x = AC(S(t)x) .$$

Proposition 3.1 in the discrete case $(D = Z^+)$ is the sufficient part of theorem 3 of [14]. In the continuous case $(D = \mathbb{R}^+)$ it is the sufficient part of theorem 2.1 of [15]. In [15] the result is stated for semigroups $S(t)$ for which $t \to S(t)x$ is continuous in t. The observation that the continuity of $S(t)x$ is not needed in this result is due to R. Schöneberg [18].

-7-

Combining (3.5) with proposition 2.6 we obtain the following result of R. Bruck [9].

Proposition 3.2

If $F \neq \phi$ then a necessary and sufficient condition for $S(t)x$ to converge weakly as $t \to \infty$ in D is

$$(3.7) \qquad w\text{-}\lim_{t \to \infty}(S(t+h)x - S(t)x) = 0 \quad \text{for all} \quad h \in D .$$

We turn now to the ergodic theorem for $S(t)x,$ that is the weak convergence of averages of $S(t)x.$ In order to state the result we consider functions

$$Q(s,t) : D \times D \to [0,\infty)$$

and assume that there is a translation invariant measure μ on D such that for every $s \in D,$ $Q(s,t) : D \to \mathbb{R}^+$ is μ measurable. In the case $D = \mathbb{R}^+,$ μ is the usual Lebesgue measure whereas in the case $D = \mathbb{Z}^+$ it is a discrete measure. We will assume that for every $s \in D,$ $Q(s,t)$ is of bounded variation and its total variation will be denoted by $V(s).$ The function $Q \in \&,$ if moreover:

$$(3.8) \qquad \int_D Q(s,t)d\mu = 1 \quad \text{for all} \quad s \in D$$

$$(3.9) \qquad \lim_{s \to \infty} \int_{D \cap [0,T]} Q(s,t)d\mu = 0 \quad \text{for every} \quad s \in D \quad \text{and} \quad T < \infty .$$

and

$$(3.10) \qquad \lim_{s \to \infty} V(s) = 0 .$$

Assuming that $t \to S(t)x$ is strongly μ-measurable and that $Q \in \&$ we define the Q-averages $\sigma(s)x$ of $S(t)x$ by

$$(3.11) \qquad \sigma(s)x = \int_D Q(s,t)S(t)x \, d\mu$$

The next lemma is the principal ingredient of the ergodic results.

Lemma 3.3

Let $S(t)$ be a semigroup of contractions on $C \subset H.$ If $F \neq \phi,$ $Q \in \&$ and $\sigma(t)x$ is defined by (3.11) then

$$(3.12) \qquad W(\sigma(t)x) \subset F \quad \text{for every} \quad x \in C .$$

-8-

Proof:

Let $x \in C$ be fixed and $y \in C$, $t, h \in D$. From

$$0 \leq \|S(t)x - y\|^2 - \|S(t+h)x - S(h)y\|^2 \leq$$

(3.13)

$$\leq \|S(t)x - S(h)y\|^2 - \|S(t+h)x - S(h)y\|^2 + 2(S(t)x - S(h)y, S(h)y - y) + \|y - S(h)y\|^2$$

it follows upon multiplying by $Q(s,t)$ and integrating over D that

$$0 \leq \int_D Q(s,t)(\|S(t)x - S(h)y\|^2 - \|S(t+h)x - S(h)y\|^2)\, d\mu$$

(3.14)

$$+ 2(\sigma(s)x - S(h)y, S(h)y - y) + \|y - S(h)y\|^2$$

Since $F \neq \phi$, $\|S(t)x - S(h)y\|^2$ is bounded and it follows from (3.9) and (3.10) that

$$\limsup_{s \to \infty} \int_D Q(s,t)(\|S(t)x - S(h)y\|^2 - \|S(t+h)x - S(h)y\|^2)\, d\mu = 0.$$

If $\sigma(s_k)$ converges weakly to p as $s_k \to \infty$, then passing to the limit through the sequence s_k in (3.14) we find

(3.15) $$0 \leq 2(p - S(h)y, S(h)y - y) + \|y - S(h)y\|^2 \quad \text{for all } y \in C.$$

From (3.8) and (3.11) it follows that $\sigma(s)x \in C$ for every $s \in D$ and therefore $p \in C$. Substituting $y = p$ in (3.15) yields $S(h)p = p$ and since $h \in D$ was arbitrary, $p \in F$. ∎

Combining lemma 3.3 with corollary 2.4 we obtain

Theorem 3.4

Let $S(t)$, $t \in D$ be a semigroup of nonexpansive mappings on $C \subset H$. If $F \neq \phi$ and $Q \in \mathcal{Q}$ then

(3.16) $$w\text{-}\lim_{t \to \infty} \sigma(t)x = p = AC(S(t)x)$$

holds for every $x \in C$.

For the case $D = \mathbb{Z}^+$ theorem 3.4 was proved by Baillon [1], in the case of Cesaro means. Simpler proofs of Baillon's result were subsequently given by L. Tartar (unpublished) and [6], [14]. The general result for $D = \mathbb{Z}^+$ is given in [6], [9], [17]. For the case $D = \mathbb{R}^+$, for Cesaro means, the theorem is given in [2]. The general case $D = \mathbb{R}^+$ was proved in [4], [17]. In these results the continuity of $t \to S(t)x$ is used in the proof. As it turns out from theorem 3.4 this continuity is not needed for

-9-

the result. In order to define $\sigma(s)x$ we had to assume that $t \to S(t)x$ is strongly

measurable on $(0,\infty)$. This implies, see [10], that $t \to S(t)x$ is continuous on

$(0,\infty)$ but not necessarily on $[0,\infty)$ as was assumed in [4], [17].

4. An ergodic theorem for products of resolvents

Let H be a real Hilbert space and let A be a maximal monotone operator in H. For the definition and elementary properties of maximal monotone operators the reader is referred to the texts [3], [5].

For $\lambda > 0$ the resolvent J_λ of A is defined by $J_\lambda = (I + \lambda A)^{-1}$ and it is well known that $J_\lambda : H \to H$ is nonexpansive. Given $x_0 \in H$ and a sequence $\{\lambda_n\} \subset \mathbb{R}^+$ we define a sequence x_n in H by:

$$(4.1) \qquad x_n = J_{\lambda_n} x_{n-1} \qquad n = 1, 2, \ldots$$

We denote by F the set $A^{-1} 0$. If $p \in F$ then $J_\lambda p = p$ for every $\lambda > 0$. Therefore, for $p \in F$ we have

$$(4.2) \qquad \|x_n - p\| = \|J_{\lambda_n} x_{n-1} - J_{\lambda_n} p\| \leq \|x_{n-1} - p\|$$

and thus

$$(4.3) \qquad F \subseteq N(\{x_n\}) \ .$$

If $F \neq \phi$ and $\lambda_n = \lambda$ for all $n = 1, 2, \ldots$ then $x_n = J_\lambda^n x_0$ and it follows from theorem 3.4 that the Cesaro means

$$(4.4) \qquad \sigma_n(x_0) = \frac{1}{n} \sum_{k=0}^{n-1} J_\lambda^k x_0$$

converge weakly as $n \to \infty$ to the asymptotic center of the sequence $\{J_\lambda^n x_0\}$. Our next proposition generalizes this result.

Proposition 4.1

Let $\{\lambda_n\} \subset \mathbb{R}^+$ be such that $\sum_{k=0}^{\infty} \lambda_k = \infty$. If x_n is the sequence defined by (4.1), $F \neq \phi$ and

$$(4.5) \qquad y_n = \Big(\sum_{k=0}^{n} \lambda_k \Big)^{-1} \sum_{k=0}^{n} \lambda_k x_k \qquad n = 0, 1, \ldots$$

then the sequence y_n converges weakly as $n \to \infty$ to a point $p \in F$. Moreover the weak limit p of y_n is the asymptotic center of the sequence $\{x_n\}$.

Proof:

Obviously $W(\{y_n\}) \subset \overline{\text{conv}} \ W(\{x_n\})$. Using corollary 2.4 it suffices to prove that $W(\{y_n\}) \subset N(\{x_n\})$. To do this we note that by (4.3) $F \subset N(\{x_n\})$ and so it suffices to show that $W(\{y_n\}) \subset F$.

-11-

Let $[\xi, \eta] \in A$. From the definition of x_n we have

(4.6) $\quad |x_n - \xi|^2 + 2\lambda_n(\eta, x_n - \xi) + |x_n - x_{n-1}|^2 \leq |x_{n-1} - \xi|^2 \qquad n = 1, 2, \ldots$

Therefore

$$2\lambda_n(\eta, x_n - \xi) \leq |x_{n-1} - \xi|^2 - |x_n - \xi|^2$$

which implies

(4.7) $\qquad\qquad 2(\eta, y_n - \xi) \leq \left(\sum_{k=0}^{n} \lambda_k \right)^{-1} \| x_0 - \xi \|^2 .$

Let $y \in W(\{y_n\})$ such that

$$\underset{n_k \to \infty}{w\text{-}\lim} \; y_{n_k} = y$$

Passing to the limit through the sequence n_k in (4.7) yields

$$(\eta, y - \xi) \leq 0 \qquad [\xi, \eta] \in A.$$

From the maximality of A we deduce $y \in A^{-1}0 = F$ and the proof is complete. ∎

Proposition 4.1 is essentially due to P. L. Lions [12]. It is interesting to know that if $\Sigma \lambda_n^2 = \infty$, the sequence $\{x_n\}$ itself converges weakly. This was proved by H. Brezis and P. L. Lions [8]. In particular in the case $\lambda_n = \lambda$, $F \neq \phi$, $J_\lambda^k x_0$ converges weakly as $k \to \infty$ and as a consequence, of course, $\sigma_n(x_0)$ converges weakly without reference to theorem 3.4. We conclude this section by proving the above mentioned result of Brezis and Lions.

Proposition 4.2

Let $\{x_n\}$ be the sequence defined by (4.1). If $F \neq \phi$ and $\sum_{n=0}^{\infty} \lambda_n^2 = \infty$ then x_n converges weakly as $n \to \infty$.

Proof:

We deduce proposition 4.2 from corollary 2.5. In view of (4.3) we have only to show that $W(\{x_n\}) \subseteq F$.

From the definition of x_n it follows that $\lambda_n^{-1}(x_n - x_{n-1}) = y_n \in Ax_n$. From the monotonicity of A we have

(4.8) $\qquad 0 \leq (y_{n+1} - y_n, x_{n+1} - x_n) = \lambda_{n+1}^{-1}(y_{n+1} - y_n, y_{n+1})$

-12-

and therefore $n \to \|y_n\|^2$ is monotone nonincreasing. Moreover, taking $\xi \in A^{-1}0$ in (4.6) yields

(4.9)
$$\lambda_n^2 \|y_n\|^2 \leq \|x_{n-1} - \xi\|^2 - \|x_n - \xi\|^2$$

and therefore by summing over n,

$$\|y_n\|^2 \sum_{k=0}^{n} \lambda_k^2 \leq \sum_{k=0}^{\infty} \lambda_k^2 \|y_k\|^2 \leq \|x_0 - \xi\|^2$$

which implies $y_n \to 0$ as $n \to \infty$. Let $x \in W(\{x_n\})$, from the monotonicity of A we have

$$(\eta - y_n, \xi - x_n) \geq 0 \qquad \forall [\xi, \eta] \in A$$

and passing to the limit as $n \to \infty$ through an appropriate subsequence we find

$$(\eta, \xi - x) \geq 0 \qquad \forall [\xi, \eta] \in A$$

which by the maximality of A implies $x \in A^{-1}0 = F$ and the proof is complete. ∎

REFERENCES

[1] J. B. Baillon, Un théorème de type ergodic pour les contractions nonlineaires dans un espace de Hilbert. C. R. Acad. Sci. Paris 280(1975) 1511-1514.

[2] J. B. Baillon and H. Brezis, Une remarque sur le comportement asymptotique des semigroupes nonlinéaires, Houston J. Math 2(1976), 5-7.

[3] V. Barbu, Nonlinear semigroups and differential equations in Banach spaces, Noordhoff, Leyden, The Netherlands 1976.

[4] H. Brezis, Asymptotic behavior of some evolution systems in "Nonlinear evolution equations" (M. G. Crandall, Ed) Academic Press (to appear).

[5] H. Brezis, Operateurs Maximaux Monotone et semigroupes de Contractions dans les Espaces de Hilbert, North-Holland, Amsterdam, 1973.

[6] H. Brezis and F. E. Browder, Nonlinear ergodic theorems, Bull. Amer. Math. Soc. 82(1976), 959-961.

[7] H. Brezis and F. E. Browder, Remarks on nonlinear ergodic theory, Adv. in Math 25(1977) 165-177.

[8] H. Brezis and P. L. Lions, Produits infinis de résolvantes, Israel Jour. Math 29(1978) 329-345.

[9] R. Bruck, On the almost-convergence of iterates of a nonexpansive mapping in Hilbert space and the structure of the weak w-limit set, Israel J. Math 29(1978) 1-16.

[10] M. Crandall and A. Pazy, Semigroups of nonlinear contractions and dissipative sets, Jour. Func. Anal. 3(1969) 376-418.

[11] M. Edelstein, The construction of an asymptotic center with a fixed point property, Bull. Amer. Math. Soc. 78(1972) 206-208.

[12] P. L. Lions, Une methode iterative de resolution d'une inequation variationnelle Israel Jour. Math (to appear).

[13] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math Soc. 73(1967) 591-597.

[14] A. Pazy, On the asymptotic behavior of iterates of nonexpansive mappings in Hilbert space, Israel Jour. Math 26(1977) 197-204.

[15] A. Pazy, On the asymptotic behavior of semigroupes of nonlinear contractions in Hilbert space, Jour. Func. Anal. 27(1978) 292-307.

[16] S. Reich, Almost convergence and nonlinear ergodic theorems, Jour. of Approx. (to appear).

[17] S. Reich, Nonlinear evolution equations and nonlinear ergodic theorems, Nonlinear Anal. Theory, Methods & Appl. 1(1977) 319-330.

[18] R. Schoeneberg, Weak convergence of trajectories of nonexpansive semigroups in Hilbert space, Israel Jour. Math (to appear).

AP/ed

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 1866	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) REMARKS ON NONLINEAR ERGODIC THEORY IN HILBERT SPACE		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) A. Pazy		8. CONTRACT OR GRANT NUMBER(s) DAAG29-75-C-0024 MCS78-01245
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below		12. REPORT DATE July 1978
		13. NUMBER OF PAGES 14
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Asymptotic center, weak w-limit set, weak convergence, fixed points

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this note we give a simple unified presentation of some recent ergodic results for semigroups of nonexpansive mappings in Hilbert space.

DD FORM 1473   EDITION OF 1 NOV 65 IS OBSOLETE
1 JAN 73